Modulation and correlations lengths in systems with competing interactions: new exponents and other universal features

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(Dated: July 1, 2009)

arXiv:0906.5381v1 [cond-mat.stat-mech] 30 Jun 2009

We examine correlation functions in the presence of competing long and short interactions to find multiple correlation and modulation lengths. For many systems, we show the presence of a crossover temperature T^* above which there is no modulation and report on a *new exponent*, ν_L, characterizing the universal nature of this divergence of the modulation length. We study large n systems and find, in general, that $\nu_L = 1/2$. We demonstrate that for a short range system frustrated by a general competing long range interaction, the crossover temperature T^* veers towards the critical temperature of the unfrustrated short range system (i.e., that in which the frustrating long range interaction is removed) . We discover in systems with long range interactions the existence of at least one diverging correlation length in the high temperature limit. When screening is present, instead of diverging, this correlation length tends, at high temperatures to the screening length. We also show that apart from certain special crossover points, the total number of correlation and modulation lengths remains conserved. We derive an expression for the change in modulation length with temperature for a general system near the ground state with a ferromagnetic interaction and an opposing long range interaction. We illustrate that the correlation functions associated with the exact dipolar interactions differ substantially from those in which a scalar product form between the dipoles is assumed.

PACS numbers: 05.50.+q, 68.60.Dv, 75.10.-b, 75.10.Hk, 75.60.Ch

I. INTRODUCTION

Short range interactions have been at the focus of much study for many decades. Perhaps one of the best known examples are the Ising ferromagnet and the antiferromagnet [1]. Long range interactions are equally abundant [2]. Systems in which both long and short range interactions co-exist comprise very interesting systems. Such competing forces can lead to a wealth of interesting patterns – stripes, bubbles, etc [3], [4], [5]. Realizations are found in numerous fields – quantum Hall systems [6], adatoms on metallic surfaces, amphiphilic systems [7], interacting elastic defects (dislocations and disclinations) in solids [8], interactions amongst vortices in fluid mechanics [9] and superconductors [10], crumpled membrane systems [11], wave-particle interactions [12], interactions amongst holes in cuprate superconductors [13], [14], [15], [16], [17], arsenide superconductors [18], manganates and nickelates [19], [20], some theories of structural glasses [21], [22], [23], [24], [25], colloidal systems [26], [27] and many many more. Much of the work to date focused on the character of the transitions in these systems and the subtle thermodynamics that is often observed (e.g., the equivalence between different ensembles in many such systems is no longer as obvious, nor always correct, as it is in the canonical short range case [28]). Other very interesting aspects of different systems have been addressed in [29].

Here we investigate the general temperature dependence of the structural features that appear in such systems when competing interactions of short and long range are present. In many such systems, there are emergent modulation lengths governing the size of various domains. We find that *these modulation lengths often adhere to various scaling laws, sharp crossovers and divergences at various temperatures* (with no associated thermodynamic transition). We also find that in such systems, correlation lengths generically evolve into modulation lengths (and vice versa) at various temperatures. The behavior of correlation and modulation lengths as a function of temperature will afford us with certain selection rules on the possible underlying microscopic interactions. In their simplest incarnation, our central results as follows:

1. In canonical systems harboring competing short (finite) and long range interactions modulated patterns appear. Depending on the type of the long range interaction, the modulation length either increases or decreases from its ground state value as the temperature is raised. We will relate this change, in lattice systems, to derivatives of the Fourier transforms of the interactions that are present.

2. There exist special crossover temperatures at which new correlation/modulation lengths come up or some cease to exist.

   (a) For many systems, there exists a sharp crossover temperature T^* at which the modulation length diverges. That is, we find that in many systems, for all temperatures larger than a threshold temperature, T^*, there are no modulations in the pair correlation functions. If for $T < T^*$, finite modulation lengths set in

then these scale as

$$L_D \sim (T^* - T)^{-\nu_L}, \tag{1}$$

irrespective of the interaction; quite universally, $\nu_L = 1/2$.

(b) In some cases, the modulation length can change by finite increment jumps instead of diverging.

(c) The total number of characteristic lengthscales (correlation + modulation) remains conserved, except at the crossover points.

3. In the high temperature limit of large n theories there generally exists one or more diverging correlation lengths, i.e.,

$$\lim_{T\to\infty} \xi = \infty. \tag{2}$$

When screening is present, the divergence is replaced by the a saturation

$$\lim_{T\to\infty} \xi = \lambda^{-1}, \tag{3}$$

where λ^{-1} is the screening length.

4. The presence of the angular dependent dipolar interaction term that frustrates an otherwise unfrustrated ferromagnet vis a vis a simple scalar product between the dipoles adds new (dominant) lengthscales. The angular dependence significantly changes the system.

We will further investigate the ground state modulation lengths in general frustrated Ising systems and also point to discontinuous jumps in the modulation length's that may appear in the large n rendition of some systems.

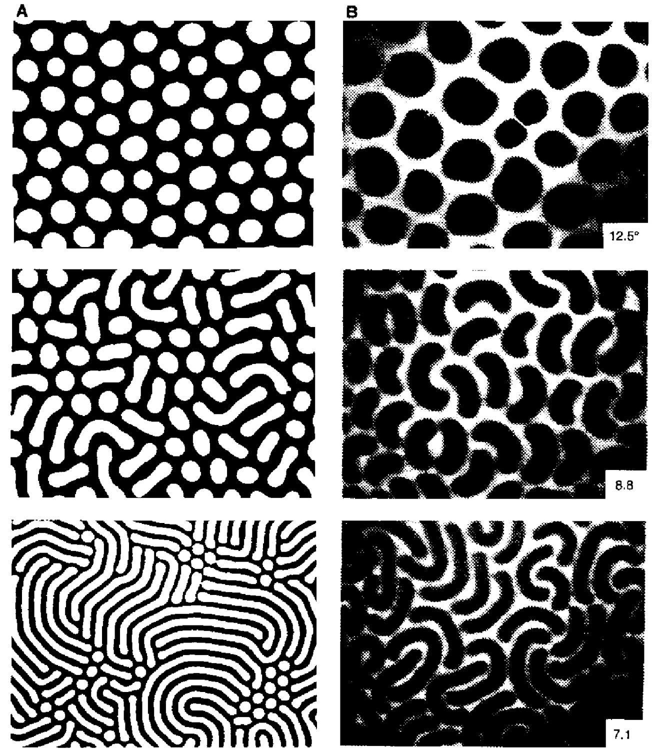

FIG. 1: Reproduced with permission from Science, Ref.[11]. Reversible "strip out" instability in magnetic and organic thin films. Period (L_D) reduction under the constraint of fixed overall composition and fixed number of domains leads to elongation of bubbles. Left panel (A) in magnetic garnet films, this is achieved by raising the temperature [labeled in (B) in degrees Celsius] along the symmetry axis, $H = 0$ (period in bottom panel, $\sim 10\mu m$) (see Fig. 5). Right panel (B) In Langmuir films composed of phospholipid dimyristolphosphatidic acid (DMPA) and cholesterol (98:2 molar ratio, pH 11), this is achieved by lowering the temperature at constant average molecular density [period in bottom panel, $\sim 20\mu m$]

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Armed with these general results, we may discern the viable microscopic interactions (exact or effective) which underlie temperature dependent patterns that are triggered by two competing interactions. Our analysis suggests the effective microscopic interactions that may drive non-uniform patterns such as those underlying lattice analogs of the systems of Fig.(1).

The treatment that we present in this work applies to lattice systems and does not account for the curvature of bubbles and other continuum objects. These may be augmented by inspecting energy functionals (and their associated free energy extrema) of various continuum field morphologies under the addition of detailed domain wall tension forms – e.g., explicit line integrals along the perimeter where surface tension exists – and the imposition of additional constraints via Lagrange multipliers. We leave their analysis for future work. One of the central results of our work is the derivation of conditions relating to the increase/decrease of modulation lengths in lattice systems with changes in temperature. These conditions relate the change in the modulation length at low temperatures to the derivatives of the Fourier transforms of the interactions present.

In Section(II), we outline the general systems that we study – short range systems that are frustrated by long range interactions. We then proceed, in Section(III) to introduce a crossover temperature T^* and prove a general scaling relation associated with it. In Section(IV), we derive the scaling form for the Ising ground states for general frustrating long range interactions. Henceforth, we provide explicit expressions for the crossover temperatures and the correlations lengths in the large n limit. In Section (V), we introduce the two spin correlation function for a general system in this limit. Based on the correlation function, we then present some general results for systems with competing nearest neighbor ferromagnetic interaction and an arbitrary long range interaction in Section (VI). We start by deriving the equilibrium stripe width for a two dimensional Ising system with

nearest neighbor ferromagnetic interactions and competing long range interactions. We derive an expression for the change in modulation length with temperature for low temperatures for large n systems. We illustrate how the crossover temperature, T^* arises in the large n limit and show some general properties of the system associated with it. We show the presence of a diverging correlation length in the high temperature limit.

We present some example systems in Section (VII). We numerically calculate the correlation function for the screened Coulomb frustrated ferromagnet and the dipolar frustrated ferromagnet. We then study the screened Coulomb frustrated ferromagnet in more details. Next, we show some results for systems with higher dimensional spins. We study a system with the dipole-dipole interaction for three dimensional spins, without ignoring the angular dependent term and show that this term changes the ground state lengthscales of the system considerably. We also present a system with the Dzyaloshinsky - Moriya interaction in addition to the ferromagnetic term and a general frustrating long range term.

We give our concluding remarks in Section (VIII).

II. THE SYSTEMS OF STUDY

Consider a translationally invariant systems whose Hamiltonian is given by

$$H = \frac{1}{2}\sum_{\vec{x},\vec{y}} V(|\vec{x}-\vec{y}|)S(\vec{x})S(\vec{y}). \tag{4}$$

The quantities $\{S(\vec{x})\}$ portray classical scalar spins or fields. The sites $\vec{x}$ and $\vec{y}$ lie on a hypercubic lattice of size N of unit lattice constant. In what follows, $v(k)$ and $s(\vec{k})$ will denote the Fourier transforms of $V(|\vec{x}-\vec{y}|)$ and $S(\vec{x})$. For analytic interactions, $v(k)$ is a function of k^2 (to avoid branch cuts). We will focus on systems with competing interactions. Throughout most of this work we will focus on the following Hamiltonian with short and long range interactions.

$$H = -J\sum_{\langle\vec{x},\vec{y}\rangle} S(\vec{x})S(\vec{y}) + Q\sum_{\vec{x}\neq\vec{y}} V_L(|\vec{x}-\vec{y}|)S(\vec{x})S(\vec{y}) \tag{5}$$

where the first term represents nearest neighbor ferromagnetic interaction for positive J and the second term represents some long range interaction which opposes the ferromagnetic interaction for positive Q. We will study properties of general systems of the form of Eq.(5). In order to flesh out the physical meaning of our results and illustrate their implications and meaning, we will further provide explicit expressions and numerical results for two particular examples. The Hamiltonian of Eq.(5) represents a system that we christen to be the screened *Coulomb Frustrated Ferromagnet* when

$$\begin{aligned} V_L(r) &= \frac{e^-\lambda r}{4\pi r} \text{ in three dimensions, and} \\ V_L(r) &= K_0(\lambda r) \text{ in two dimensions,} \end{aligned} \tag{6}$$

where λ^{-1} represents the screening length and K_0 is a modified Bessel function,

$$K_0(x) = \int_0^\infty dt \frac{\cos xt}{\sqrt{1+t^2}}. \tag{7}$$

Throughout our work, we will discuss both the screened and unscreened renditions of the Coulomb frustrated ferromagnet. Eq.(5) corresponds to a *Dipolar Frustrated Ferromagnet* when

$$\begin{aligned} V_L(r) &= \frac{1}{r^3} \\ &= \frac{1}{(r^2+\delta^2)^{3/2}} \text{ in the limit } \delta \to 0, \end{aligned} \tag{8}$$

on the lattices that we will consider. Later, we also consider the general direction dependent (relative to the location vectors) of the dipolar interaction for three dimensional spins; we will replace the scalar product form of the dipolar interactions in Eqs.(5, 8) by the precise dipolar interactions between magnetic moments.

On a hypercubic lattice, the nearest neighbor interactions in real space of Eq.(5) have the lattice Laplacian

$$\Delta(\vec{k}) = 2\sum_{l=1}^{d}(1-\cos k_l), \tag{9}$$

as their Fourier transform. In the continuum (small k) limit, $\Delta \to z = k^2$. The real lattice Laplacian

$$\langle\vec{x}|\Delta|\vec{y}\rangle = \begin{cases} 2d & \text{for } \vec{x}=\vec{y} \\ -1 & \text{for } |\vec{x}-\vec{y}|=1. \end{cases} \tag{10}$$

Notice that $\langle\vec{x}|\Delta^R|\vec{y}\rangle = 0$ for $|\vec{x}-\vec{y}| > R$, where R is the spatial range over which the interaction kernel is non-vanishing. Eq.(10) corresponds to interactions of Range=2 lattice constants,

$$\begin{aligned} \langle\vec{x}|\Delta^2|\vec{y}\rangle = \quad & 2d(2d+2) \text{ for } \vec{x}=\vec{y} \\ & -4d \text{ for } |\vec{x}-\vec{y}|=1 \\ & 2 \text{ for } (\vec{x}-\vec{y}) = (\pm\hat{e}_\ell \pm \hat{e}_{\ell'}) \text{ where } \ell\neq\ell' \\ & 1 \text{ for a } \pm 2\hat{e}_\ell \text{ separation.} \end{aligned} \tag{11}$$

Correspondingly, in the continuum, the Lattice Laplacian and its powers attain simple forms and capture tendencies in numerous systems. Surface tension in many systems is captured by a $g(\nabla\phi)^2$ term where ϕ is a constant in a uniform domain. Upon Fourier transforming, such squared gradient terms lead to a k^2 dependence. The effects of curvature which are notable in many mixtures and membrane systems can often be emulated by terms involving $(\nabla^2 h)$ with h a variable parameterizing the profile; at times the interplay of such curvature terms with others leads, in the aftermath, to a simple short range k^4 term in the interaction kernel (the continuum version of the squared lattice Laplacian of Eq.(11)). An excellent review of these issues is addressed in [11].

III. A UNIVERSAL DOMAIN LENGTH EXPONENT

We next report on a new exponent for the domain length of general frustrated systems. Our result below applies to *general fields*. It spans real or complex scalar fields, vectorial (or tensorial) fields of both the discrete (e.g., Potts like) and continuous variants. Although the result below is general, for consistency of notation in this work, we will refer to a system of $O(n)$ spins. The designation of "$O(n)$ spins" simply denotes real fields (spins) of unit length that have some arbitrary number (n) of the components. For $n = 1$, the system is an Ising model. A single component real field having unit norm allows for only two scalars at any given site $\vec{x}$: $S(\vec{x}) = \pm 1$. The $n = 2$ system corresponds to a two component spin system in which the spins are free to rotate in a two dimensional place- $S_1^2(\vec{x}) + S_2^2(\vec{x}) = 1$ (the so-called XY spin system). The case of $n = 3$ corresponds to a system of three component Heisenberg type spins, and so on. In general,

$$1 = \sum_{a=1}^{n} S_a(\vec{x}) S_a(\vec{x}). \tag{12}$$

In the up and coming, we will denote the momentum space correlator by

$$G(\vec{k}) = \langle \vec{S}(\vec{k}) \cdot \vec{S}(-\vec{k}) \rangle.) \tag{13}$$

In the case of the Ising model, $n = 1$, the scalar product in Eq.(13) is to be replaced by a product between the complex scalars $S(\vec{k})$ and $S(-\vec{k})$. The characteristic lengthscales of the system are given by the solutions of $G^{-1}(T, k) = 0$ where $G^{-1}(T, k)$ denotes the inverse pair correlator at wavenumber k and temperature T. We will now consider the general case of Eq.(4) in which the system exhibits modulations at a commensurate wavenumber q for all temperatures $T > T^*$, and starts to exhibit incommensurate modulations at temperatures $T = T^{*-}$. Throughout this work, we will provide several specific examples. Whenever the above conditions are satisfied (i.e., incommensurate modulations set in below a temperature T^*) then the modulation must universally follow the power law scaling as a function of the temperature $(T^* - T)$ with an exponent of $\nu_L = 1/2$. We first show how this result occurs generally.

As the system exhibit modulations of wavenumber q above T^*, the real part of the poles of the correlator must be q. That is,

$$G^{-1}(T, q + i\kappa) = 0. \tag{14}$$

If plot the curve for Eq.(14) in the $T - \kappa$ plane, T^* will denote the point where the curve just touches the line $T = T^*$ from above. The curve is parallel to the κ-axis at that point. Thus,

$$\frac{\partial G^{-1}}{\partial k}\Big|_{(T^*, q+i\kappa)} = 0. \tag{15}$$

We investigate the case in which $T^* \neq T_c$ and for which the inverse correlator is analytic at T^* and an expansion is possible. For poles at a slightly lower temperature, we have to lowest order,

$$\begin{aligned} & G^{-1}(T, k) = 0 \\ \Longrightarrow \quad & \Big[(T^* - T)\frac{\partial G^{-1}}{\partial T}\Big|_{(T^*, q+i\kappa)} \\ & + \frac{\delta k^2}{2!}\frac{\partial^2 G^{-1}}{\partial k^2}\Big|_{(T^*, q+i\kappa)}\Big] = 0. \end{aligned} \tag{16}$$

Thus, if δk is not imaginary then, to leading order,

$$\left|\frac{2\pi}{L_D} - q\right| \propto (T^* - T)^{\nu_L},$$

$$\text{with} \qquad \boxed{\nu_L = \frac{1}{2}}. \tag{17}$$

Eq.(17) constitutes a new counterpart for the well known divergence of the correlation length at T_c. There is now a new divergence of the modulation length at the "disorder" line temperature T^*. [30] The value of the critical exponent is similar to that appearing for the correlation length exponent for mean-field or large n theories. It should be stressed that our result of Eq.(17) is far more general.

IV. GROUND STATE STRIPE WIDTH FOR ISING SYSTEMS

We next briefly address the opposite extreme – low temperatures, and discuss the ground states of general long range interactions in Eq.(5). Below, we discuss the Ising ground states. We will later on consider the spherical model that will enable us to compute the correlation functions at arbitrary temperatures. We consider a system with Ising spins in d dimension and assume that the system forms a "striped" pattern (periodic pattern along one of the dimensions – stripes in two dimensions, planes that lie in three dimensions and so on) of spin-up and spin-down states of period l. We then calculate the free energy as a function of l and then minimize it with respect to l to get the equilibrium stripe width. For the frustrated ferromagnet, if $v_L(k) = 1/k^p$,

$$l = \left[\frac{(2\pi)^{p+2} J}{4Qp\left(1 - \frac{1}{2^{p+2}}\right)\zeta(p+2)}\right]^{\frac{1}{p+1}}, \tag{18}$$

where $\zeta(s)$ is the Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{19}$$

For Coulomb frustration, $p = 2$,

$$l = \sqrt[3]{\frac{64\pi^2}{5}\frac{J}{Q}}, \tag{20}$$

in accord with the results of Refs. [23, 31]. For long range dipolar interactions ($p = 3$), we find that

$$l = \delta\sqrt{\frac{3J}{Q}}. \tag{21}$$

In the notation to be employed throughout this work, l plays the role of the modulation length of the system at zero temperature, $L_D(T = 0)$.

V. CORRELATION FUNCTIONS IN THE LARGE N LIMIT - GENERAL CONSIDERATIONS

The results reported henceforth were computed within the spherical or large n limit [32]. It was found by Stanley long ago that the large n limit of the n component normalized spin systems (so called $O(n)$ spins) introduced in Section(III) is identical to the spherical model first introduced by Berlin and Kac. [32] We now introduce this model in its generality. The spins in Eq.(4) satisfy a single global ("spherical") constraint,

$$\sum_{\vec{x}} \langle S^2(\vec{x})\rangle = N, \tag{22}$$

enforced by a Lagrange multiplier μ. This leads to the functional $H' = H + \mu N$ which renders the model quadratic (as both Eqs.(4, 22) are quadratic) and thus exactly solvable, see, e.g., [14]. The continuum analogs of Eqs.(4,22) read

$$H = \frac{1}{2}\int d^dx\ d^dy\ V(|\vec{x}-\vec{y}|)S(\vec{x})S(\vec{y}),$$
$$\int d^dx\ \langle S^2(\vec{x})\rangle = const. \tag{23}$$

The quadratic theory may be solved exactly. From the equipartition theorem, for $T \geq T_c$, the Fourier space correlator

$$G(k) = \langle |S^2(k)|\rangle = \frac{k_BT}{v(k)+\mu}. \tag{24}$$

The real space two point correlator is given by

$$G(\vec{x}) \equiv \langle S(0)S(\vec{x})\rangle = k_BT\int_{BZ} \frac{d^dk}{(2\pi)^d}\frac{e^{i\vec{k}\cdot\vec{x}}}{v(k)+\mu}, \tag{25}$$

with d the spatial dimension and BZ denoting the integration over the first Brillouin zone. For a hypercubic lattice in d dimensions with a lattice constant that is set to one, $-\pi < k_i \leq \pi$ for $i = 1, 2, ..., d$. Henceforth, to avoid cumbersome notation, we will generally drop the designation of BZ; this is to be understood on all momentum space integrals pertaining to the lattice systems that we examine. To complete the characterization of the correlation functions at different temperatures, we note that the Lagrange multiplier $\mu(T)$ is given by the implicit equation $1 = G(\vec{x} = 0)$. Thus,

$$1 = k_BT\int \frac{d^dk}{(2\pi)^d}\frac{1}{v(k)+\mu}. \tag{26}$$

This implies that the temperature T is a monotonic increasing function of μ. If μ changes by a small amount $\Delta\mu$, then T will change by an amount ΔT, such that

$$\Delta T \propto \Delta\mu. \tag{27}$$

Eq.(26) implies that in the high temperature limit,

$$\frac{\mu}{k_BT} = \frac{\pi^{d/2}\Lambda^d}{(2\pi)^d\Gamma(\frac{d}{2}+1)} \tag{28}$$
$$\implies T \propto \mu, \tag{29}$$

where Λ represents the upper limit of the k integration, representing the ultra-violet cut-off. If we perform the momentum integration in a hypercube of side 2π, we have in the high temperature limit,

$$\mu = k_BT. \tag{30}$$

We investigate the general character of the correlation functions given by Eq.(25) for rotationally invariant systems. If the minimum (minima) of $v(k)$ occur(s) at momenta q far from the Brillouin zone boundaries of the cubic lattice then we may set the range of integration in Eq.(25) to be unrestricted. The correlation function is then dominated by the location of the poles (and/or branch cuts) of $1/[v(k) + \mu]$. Thus, we look for solutions to the following equation.

$$v(k) + \mu = 0. \tag{31}$$

The system is perfectly ordered in its ground state. From a temperature at which the system is not perfectly ordered, as we lower the temperature, the correlation length diverges at $T = T_c$. For $T = T_c$, μ takes the value,

$$\mu_{min} = -\min_{k\in BZ}[v(k)]. \tag{32}$$

As the temperature is increased, the disorder creeps in and in most systems, at a temperature T^*, the modulation length diverges.

The characteristic length scales of the system are governed by the poles of $[v(k) + \mu]^{-1}$.

$$J\Delta(\vec{k}) + Qv_L(k) + \mu = 0, \tag{33}$$

which in the continuum limit takes the form

$$Jk^2 + Qv_L(k) + \mu = 0. \tag{34}$$

Employing the above considerations, we will derive, in the next section, some general results for systems of the form Eq.(5).

Our work will focus on classical systems. The extension to the quantum arena [14] is straightforward. In, e.g., large n bosonic renditions of our system, replicating the usual large n analysis, [24] we find that the pair correlator

$$G_B(\vec{k}) = \frac{n_B(\sqrt{\frac{v(\vec{k})+\mu}{k_B T}}) + \frac{1}{2}}{\sqrt{\frac{v(\vec{k})+\mu}{k_B T}}}, \quad (35)$$

with the bosonic distribution function

$$n_B(\frac{x}{k_B T}) = [\exp(\frac{x}{k_B T}) - 1]^{-1}. \quad (36)$$

The correlator of Eq.(35) is of a similar nature as that of the classical correlator of Eq.(24) with branch cuts generally appearing in the quantum case. Our analysis below relies on the evolution of the poles of $v(k) + \mu$ as a function of temperature in classical systems. In the quantum arena, these poles are replaced by branch points and the analysis remains qualitatively the same.

VI. RESULTS

In this section, we present some general results for systems of the form Eq.(5) in their large n limit. First, we find the dependence of the modulation length on temperature, near T_c. Next, we will illustrate an analogy between the behavior of the correlation length near the critical temperature T_c and that of the modulation length near T^*. We will then discuss some aspects of the crossover points. We end this section with some results in the high temperature limit.

A. The low temperature limit: a criterion for determining an increase or decrease of the modulation length at low temperatures

In this section, we derive universal conditions for increasing or decreasing modulation lengths in systems with competing ferromagnetic and long range interactions. Eqs.(47, 49) show general conditions for the change in modulation length, L_D with temperature for a general system of the form Eq.(5). The value, k_0 of k which satisfies Eq.(32),

$$v(k_0) = \min_{k \in BZ} v(\vec{k}) \quad (37)$$

determines the modulation length at $T = T_c$.

$$J k_0^2 + Q v_L(k_0) + \mu_{min} = 0, \quad (38)$$
$$2 J k_0 + Q v_L'(k_0) = 0. \quad (39)$$

As the temperature is raised, the new pole near k_0 will have an imaginary part corresponding to the finite correlation length. The real part will also change in general and this would induce a change in the modulation length. Let $\mu = \mu_{min} + \delta\mu$ with $\delta\mu > 0$. Then we have,

$$k = k_0 + \delta k, \qquad \delta k = \sum_{j=1}^{\infty} \delta k_j, \quad (40)$$

where $\delta k_j \propto \delta\mu^{x_j}$, $x_{j+1} > x_j$. Our goal is to find the leading order real contribution to δk which would give us the change in modulation length with increasing $\delta\mu$ and hence with increasing temperature.

$$J\delta k^2 + Q \sum_{j=2}^{\infty} v_L^{(j)}(k_0) \frac{\delta k^j}{j!} + \delta\mu = 0. \quad (41)$$

Suppose $v_L^{(n)}(k_0) = 0$ for $2 < n < p$ and $v_L^{(p)}(k_0) \neq 0$. We have,

$$(J + \frac{Q v_L^{(2)}(k_0)}{2!})(\delta k_1^2 + 2\delta k_1 \delta k_2 + ...) + [\frac{Q v_L^{(p)}(k_0)}{p!}(\delta k_1^p + p\delta k_1^{p-1}\delta k_2 + ...) + \frac{Q v_L^{(p+1)}(k_0)}{(p+1)!} \times (\delta k_1^{p+1} + (p+1)\delta k_1^p \delta k_2 + ...) + ...] + \delta\mu = 0. \quad (42)$$

To leading order,

$$(J + \frac{Q v_L^{(2)}(k_0)}{2!})\delta k_1^2 + \delta\mu = 0, \qquad \delta k_1^2 = -\frac{\delta\mu}{J + \frac{Q v_L^{(2)}(k_0)}{2!}}. \quad (43)$$

From this, we see that δk_1 is imaginary. This constitutes another verification of the well established result about the universality of the divergence of the correlation length, ξ at T_c with the mean-field type critical exponent $\nu = 1/2$ in the large n limit.

$$\xi \propto (T - T_c)^{-\nu}, \qquad \nu = \frac{1}{2}. \quad (44)$$

The next, higher order, relations are obtained using the method of dominant balance.

$$2(J + \frac{Q v_L^{(2)}(k_0)}{2!})(\delta k_1)(\delta k_2) + \frac{Q v_L^{(p)}(k_0)}{p!}(\delta k_1)^p = 0$$
$$\delta k_2 = \frac{(-1)^{\frac{p+1}{2}} Q v_L^{(p)}(k_0) (\delta\mu)^{\frac{p-1}{2}}}{2p!(J + \frac{Q v_L^{(2)}(k_0)}{2!})^{\frac{p+1}{2}}}. \quad (45)$$

Therefore, δk_2 is real if p is odd and imaginary if p is even. If,

$$L_D(T) = L_D(T_c) + \delta L_D, \quad (46)$$

then, for $p = 2n + 1$,

$$\delta L_D = \frac{2\pi}{k_0^2} \frac{(-1)^n Q v_L^{(2n+1)}(k_0) \delta\mu^n}{2(2n+1)!(J + \frac{Q v_L^{(2)}(k_0)}{2!})^{n+1}}. \quad (47)$$

Thus to get the leading order real contribution to δk for even $p[>2]$, we have to go to higher order.

$$2(J+\frac{Qv_L^{(2)}(k_0)}{2!})\delta k_1\delta k_3+\frac{Qv_L^{(p+1)}(k_0)}{(p+1)!}\delta k_1^{p+1}=0$$
$$\delta k_3=\frac{(-1)^{1+p/2}Qv_L^{(p+1)}(k_0)(\delta\mu)^{p/2}}{2(p+1)!(J+\frac{Qv_L^{(2)}(k_0)}{2!})^{p/2+1}}. \quad (48)$$

For $p=2n$,

$$\delta L_D=\frac{2\pi}{k_0^2}\frac{(-1)^nQv_L^{(2n+1)}(k_0)(\delta\mu)^n}{2(2n+1)!(J+\frac{Qv_L^{(2)}(k_0)}{2!})^{n+1}}. \quad (49)$$

If, for $p=2n$, $v_L^{(2n+1)}(k_0)=0$, then we will need to continue this process until we get a real contribution to δk. In appendix B, we provide explicit forms for δL_D for different values of p. *The results derived above allow us to relate an increase/decrease in the modulation length at low temperatures to the sign of the first non-vanishing derivative (of an order larger than two) of the Fourier transform of the longer range interactions that are present.* It is important to emphasize that our results apply to any interaction, v_L that augments a nearest neighbor type interaction. These may include screened or unscreened Coulomb and other long range interactions but may also include interactions that are strictly of finite range (e.g., next-nearest neighbor interactions on the lattice for which we have $v_L=-t\Delta^2$ (with a constant $t>0$, see Eq.(11))).

The results from this section about modulation lengths, can give us similar behavior of the correlation lengths at temperatures slightly below T^*.

B. A correspondence between the temperature T^* at which the modulation length diverges and the critical temperature T_c

The critical temperature T_c corresponds to the maximum value of μ for which Eq.(31) still attains a real solution. Thus,

$$\begin{aligned} v(k_0)+\mu_{min}&=0,\\ v'(k_0)&=0,\\ v''(k_0)&>0. \end{aligned} \quad (50)$$

For systems in which the modulation length diverges at T^*, T^* corresponds to the minimum value of μ for which Eq.(31) has a purely imaginary solution. Thus, if $v(i\kappa)=\hat{v}(\kappa)$,

$$\begin{aligned} \hat{v}(\kappa_0)+\mu^*&=0\\ \hat{v}'(\kappa_0)&=0\\ \hat{v}''(\kappa_0)<0 \implies v''(i\kappa_0)&>0 \end{aligned} \quad (51)$$

Thus, we expect similar qualitative results for the correlation lengths at temperatures slightly above T_c as for modulation lengths slightly below T^* and vice-versa. [The relations for the derivatives of $\hat{v}(\kappa_0)$ in Eq.(51) are guaranteed to hold only if $T^*>T_c$.]

C. Crossover temperatures: Emergent modulations

For systems with competing multiple range interactions, there may exist special temperatures at which the poles of the correlation function change character, thus changing modulation lengths to correlation lengths and vice-versa. In particular, for most systems we have a crossover temperature T^* above which the system does not have any modulation. Apart from this kind of phenomenon, there might also be finite discontinuous jumps in the modulation length. This is illustrated with an example in Section (VI E).

We start by defining the crossover temperature T^* for a ferromagnetic system frustrated by a general long range interaction. Let $k=i\kappa$, $\kappa\in\mathbb{R}$ above T^* and $\kappa=\kappa_0$ at T^*. Let $v(k)=f(z)$, $z=k^2$. Above T^*, $\mu=\mu_{min}+\Delta\mu$ $[\Delta\mu>0]$. Using Eq.(32),

$$\begin{aligned} \mu=-f(-\kappa^2) &= -\min_{k\in\mathbb{R}} v(k)+\Delta\mu\\ &= \max_{k\in\mathbb{R}}[-v(k)]+\Delta\mu. \end{aligned} \quad (52)$$

T^* corresponds to the minimum value of $\Delta\mu$ for which we have at least one such solution. See Fig.(2). Thus,

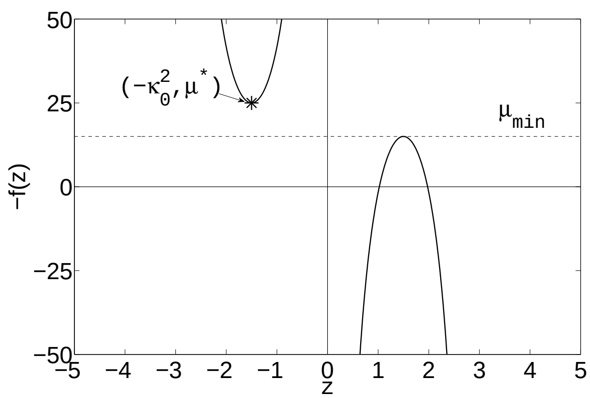

FIG. 2: $-f(z)=-v(k)=\mu$ plotted against $z=k^2$ for purely real and purely imaginary k's ($T\to 0$ and $T\to\infty$ respectively). The negative z regime corresponds to temperatures (Lagrange multiplier, μ's) for which purely imaginary solutions exist. The maximum of the curve in the positive z regime corresponds to the modulations at $T=T_c$ $[\mu=\mu_{min}]$, which is the maximum temperature at which pure modulations exist.

$$\begin{aligned} \mu^* &= \min_{\substack{\kappa\in\mathbb{R}\\ -v(i\kappa)\geq\mu_{min}}} [-v(i\kappa)]\\ &= -\max_{\substack{\kappa\in\mathbb{R}\\ -v(i\kappa)\geq\mu_{min}}} [v(i\kappa)]. \end{aligned} \quad (53)$$

Sometimes, the crossover point is slightly more difficult to visualize. See Fig.(3). In this case, the minimum upper branch of $-f(z)$ for $z < 0$ [equivalently the upper branches of $-v(i\kappa)$] gives us the value of μ^*. The branch chosen has to continue to $\mu = +\infty$ so that at least one term without modulation is always available as we increase the temperature, as required by the definition of T^*. The other branch provides such solutions only up to a certain temperature. Also, the part of it which is below μ_{min} is irrelevant.

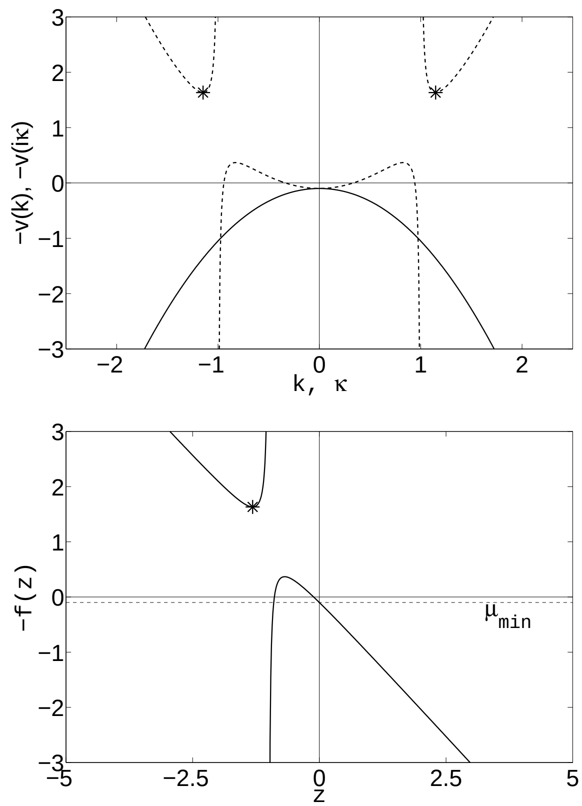

FIG. 3: TOP: Solid line:$-v(k)$ plotted against k.
Dashed line:$-v(i\kappa)$ plotted against κ.
BOTTOM: $-f(z)$ plotted against z.

If $f(z)$ is an odd function of z (e.g. the Coulomb frustrated ferromagnet), $\mu^* = -\mu_{min}$ and the correlation length at T^* is the same as the modulation length at T_c.

Also, for the system in Eq.(5), if $\lim_{\kappa\to 0} v_L(i\kappa) = +\infty$, we have, $\mu^* = \mu_{min}$, and $T^* = T_c$.

1. *$T^* = T_c$ if all the competing interactions are of finite range and crossover exists*

For systems where all the competing interactions are of finite range, $T^* = T_c$. We prove this as follows. Since finite range interactions contribute to $v(k)$ as powers of $\Delta(k) \to k^2$, for a system with only finite range interactions, $f(z)$ is analytic for all z. For a minimum of $-f(z)$ to exist in the $z < 0$ regime which is higher than the maximum in the $z > 0$ regime, we need $f(z)$ to be discontinuous at some point. Putting all of the pieces together, we find that there are no possibilities: (i) no crossover, i.e., $T^* = \infty$ or (ii) $\kappa_0 = 0$ and $\mu^* = \mu_{min}$ with $T^* = T_c$.

2. *$T^* \to T_c$ as the strength of the long range interaction is turned off*

The results from this section and the next hold for a general system, not just the frustrated ferromagnet.

The crossover temperature T^* tends to T_c for $Q = 0$ as $Q \to 0$. For a general system, let $G(T, k)$ denote the Fourier space correlation function at temperature T. By definition, at $T = T_c$ the correlation length is infinity. Thus, T_c is the solution to

$$G^{-1}(T, k) = 0, \tag{54}$$

such that $k \in BZ$ (or for continuum renditions, $k \in \mathbb{R}$).

T^* is the temperature at which the modulation length diverges for the frustrated ferromagnet, or becomes the same as the modulation length of the unfrustrated system at T_c for a general system. Thus, T^* is the solution to

$$G^{-1}(T, q + i\kappa) = 0, \tag{55}$$

with $\kappa \in \mathbb{R}$ [$q = 0$ for the case of the frustrated ferromagnet, $q = \pi$ for the frustrated antiferromagnet]. At T_c, for $Q = 0$, we have,

$$G^{-1}(T_c, q) = 0. \tag{56}$$

This however also satisfies Eq.(55), with $\kappa = 0$. Therefore,

$$\lim_{Q\to 0} T^* = T_c. \tag{57}$$

We demonstrate this in the large n limit. See Fig.(4). For $Q = 0$, we have $\mu_{min} = 0$ and $k_0 = 0$. Let $v_L(k)$ diverge as k^{-2p} near $k = 0$. For small Q, from Eq.(32), we have,

$$k_0 = (\frac{pQ}{J})^{\frac{1}{2p+2}},$$
$$\mu_{min} = -\frac{p+1}{p^{\frac{p}{p+1}}} J^{\frac{p}{p+1}} Q^{\frac{1}{p+1}}. \tag{58}$$

If p is odd,

$$\mu^* = -\mu_{min},$$
$$\kappa_0 = k_0. \tag{59}$$

As $Q \to 0$, $\kappa_0 = k_0 = 0$ and $\mu^* = \mu_{min} = 0$, that is,

$$\lim_{Q\to 0} T^* = T_c(Q = 0). \tag{60}$$

3. Proof of the conservation of the total number of characteristic lengthscales

In this section we consider the general situation in which the Fourier transform of the interaction kernel, $v(k)$, is a rational function of $z = \Delta(k)$, [$z \to k^2$ in the continuum limit]. That is, we consider situations for which

$$v(k) = f(z) = \frac{P(z)}{Q(z)}, \tag{61}$$

with P and Q polynomials (of degrees M_1 and M_2 respectively). We will now demonstrate that the combined sum of the total number of correlation and the number of modulation lengths remains unchanged as the temperature is varied.

Before providing the proof of our assertion, we first reiterate that the form of Eq.(61) is rather general. For a system with finite range interactions ($V(|\vec{x}-\vec{y}| > R) = 0$ with finite R) that is even under parity ($V(\vec{x}-\vec{y}) = V(\vec{y}-\vec{x})$), the Fourier transform of $V(\vec{x}-\vec{y})$ can be written as a finite order polynomial in $(1-\cos k_l)$ with the spatial direction index $1 \le l \le d$ where d is the dimensionality. In the small $|\vec{k}|$ (continuum limit), $[1-\cos k_l] \to k_l^2/2$. The particular case of a system with only finite range interactions that exist up to a specified range R on the lattice (the range being equal to a graph distance measuring the number of lattice steps beyond which the interactions vanish) of the form of Eq.(61) corresponds to $v(k) = P(z)$ with the order of the polynomial M_1 being equal to the interaction range, $R = M_1$. Our result below includes such systems as well as general systems with long range interactions. For long range interactions such as, e.g., the screened Coulomb frustrated ferromagnet, $f(z) = 1/(z+\lambda^2)$. The considerations given below apply to the correlations along any of the spatial directions l (and as a particular case, radially symmetric interactions for which the correlations along all directions attain the same form).

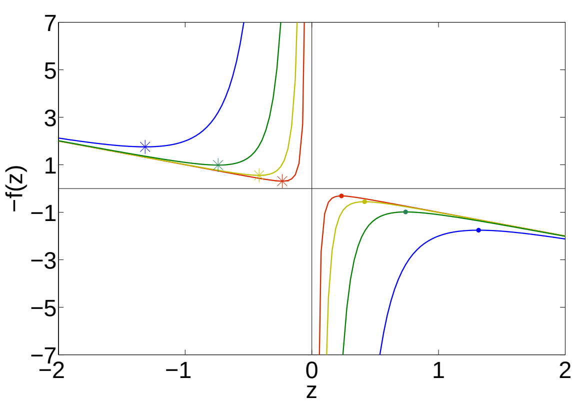

FIG. 4: Illustration of the limit $T^* \to T_c$ as $Q \to 0$ with $v_L(k) = 1/k^3$. The plot shows $-f(z) = -v(k)$ vs $z = k^2$, for $v(k) = Jk^2 + Q/k^3$ with $J = 1$ and $Q = \{1-$Blue, $0.1-$Green, $0.01-$Yellow, $0.001-$Red$\}$. '*' represents the value of μ^* and dot represents μ_{min}.

Returning to the form of Eq.(61), the Fourier space correlator of Eq.(24) is given by

$$G(\vec{k}) = k_B T \frac{Q(z)}{F(z)}; \quad F(z) = P(z) + \mu Q(z). \tag{62}$$

On Fourier transforming Eq.(62) to real space to obtain the correlation and modulation lengths, we see that the zeros of $F(z)$ determine these lengths. Expressed in terms of its zeros, F can be written as

$$F(z) = A \prod_{j=1}^{M} (z - z_j), \tag{63}$$

where $M = \max[M_1, M_2]$. Perusing Eq.(62), we see that F is a polynomial in z with real coefficients. As $F^*(z) = F(z^*)$ it follows that all roots of F are either (a) real or (b) come in complex conjugate pairs ($z_j = z_i^* \neq z_i$). We now focus on the two cases separately.

(a) Real roots: If a particular root $z_j = a^2$, $a \in \mathbb{R}$ then on Fourier transforming Eq.(62) by the use of the residue theorem, we obtain a term with a modulation length, $L_D = 2\pi/a$. Conversely, if $z_j = -a^2$, we get a term with a correlation length, $\xi = 1/a$.

(b) Next we turn to the case of complex conjugate pairs of roots. If the pair of roots z_j, z_j^* is not real, that is, $z_j = |z_j| e^{i\theta}$, then on Fourier transforming, we obtain a term containing both a correlation length, $\xi = (\sqrt{|z_j|}|\sin\frac{\theta}{2}|)^{-1}$, and modulation length, $L_D = 2\pi(\sqrt{|z_j|}|\cos\frac{\theta}{2}|)^{-1}$.

Putting all of the pieces together see that as (a) each real root of $F(z)$ contributes to either a correlation length or a modulation length and (b) complex conjugate pairs of non-real roots contribute to one correlation length and one modulation length,*the total number of correlation and modulation lengths remains unchanged as the temperature (μ) is varied.* The total number of correlation + modulation lengths is given by the number of roots of $F(z)$ (that is, M). Thus, the system generally displays a net of M correlation and modulation lengths. This concludes our proof. At very special temperatures, the Lagrange multiplier $\mu(T)$ may be such that several poles degenerate into one – thus lowering the number of correlation/modulation lengths at those special temperatures. Also, in case $M = M_2$, the total number of roots drops from M_2 to M_1 at $\mu = 0$. What underlies multiple length scales is the existence of terms of different ranges (different powers of z in the illustration above) – not frustration.

The same result can be proven using the transfer matrix method, for a one-dimensional system with Ising spins. This is outlined in appendix A. A trivial extension enables similar results for other discrete spin systems (e.g., Potts spins).

D. The high temperature limit

In this section, we show that in the high temperature limit, in systems with competing short and long range interactions, there exists the usual correlation length which goes to zero. Apart from this, we show that there also exists at least another correlation length which diverges or tends to the screening length if screening is present. The correlation function however satisfies,

$$\lim_{T\to\infty} G(\vec{x}\neq 0) = 0. \tag{64}$$

Thermodynamically, the pre-factor multiplying a term with diverging correlation length should go to zero in the high temperature limit and it indeed does.

The results in this section are valid for an arbitrary long range frustrating interaction $V_L(r)$, in the large n limit.

The correlation and modulation lengths at the high temperature is governed by the roots of Eq.(34), for large μ. The long range interaction usually satisfies,

$$\lim_{k\to 0} v_L(k) = \infty, \tag{65}$$

$$\lim_{k\to\infty} v_L(k) = 0. \tag{66}$$

We look for self-consistent solutions to Eq.(34).

1. Small correlation length – The usual behavior

Small correlation lengths correspond to a large $|k|$ of the poles of the pair correlator in Fourier space (or more precisely, to a large imaginary part of the poles in k space). We now look for self-consistent solution to the equation $G^{-1}(k) = 0$ that have large $|k|$ and for which $|k|^2 \gg |v_L(k)|$ and because the temperature is high and the system has a high value of μ also satisfy $\mu \gg |v_L(k)|$. With all of the above in tow, the pole equation $G^{-1}(k) = 0$ reduces to the equation obtained in the "canonical" case of short range interactions,

$$\begin{aligned} Jk^2 &= -\mu, \\ k &= i\sqrt{\frac{\mu}{J}}. \end{aligned} \tag{67}$$

Thus, at least one correlation length is, to leading order,

$$\xi = \sqrt{\frac{J}{\mu}}. \tag{68}$$

Eq.(29) states that in the high temperature limit, in any dimension,

$$\lim_{T\to\infty} \xi \propto \frac{1}{\sqrt{T}}. \tag{69}$$

Such a behavior is anticipated; at high temperatures, the exhibits correlations only up to small distances.

2. A new divergence or saturation (in presence of screening) of the correlation length

We search for a self-consistent solution with a small modulus $|k| \ll 1$ to the equation $G^{-1}(k) = 0$ in the limit of high temperatures (high μ). A small value of $|k|$ mandates a small value of the imaginary part of k and thus to a large correlation length. Setting $|v_L(k)| \gg |k|^2$ (and $\mu \gg |k|^2$) we find, to leading order, that $G^{-1}(k) = 0$ reads

$$\begin{aligned} Qv_L(k) &= -\mu, \\ k &= v_L^{-1}(\frac{\mu}{Q}), \end{aligned} \tag{70}$$

where, v_L^{-1} is the inverse function of v_L. That is, $v_L^{-1}(v_L(q)) = q$. From Eq.(65), we find that the correlation length,

$$\xi_{T\to\infty} \to \infty. \tag{71}$$

We next briefly comment on the momentum space kernel

$$v_L(k) = \frac{1}{(k^2+\lambda^{-2})^{p/2}} \tag{72}$$

that corresponds to a screened variant of $V(r) \sim \frac{1}{r^x}$ with $x < d$ for $p > 0$ with a screening length λ^{-1} which we take to be small. In the high temperature limit, we find a self-consistent small $|k|$ solution to the pole equation $G^{-1}(k) = 0$. This solution illustrates that

$$\lim_{T\to\infty} \xi = \lambda^{-1}. \tag{73}$$

That is, at least one of the correlation lengths tends to the screening length λ^{-1} in the limit of high temperatures. In the absence of screening, i.e., $\lambda^{-1} = 0$, at least one correlation length diverges in the high temperature limit! This might seem paradoxical. However, a careful evaluation shows that in this limit the prefactor associated with this correlation length tends to zero as $T \to \infty$ and all correlations decay monotonically with increasing temperature as they must. The reader can see how this occurs in some of the example systems that we will examine in detail later on (e.g., Eqs.(85, 86)).

E. First order transitions in the modulation length

In this section, we show that there might be systems in which the modulation length makes finite discontinuous jumps. In these situations, the modulation length does not diverge at a temperature T^* (or set of such temperatures). The ground state modulation lengths (the reciprocals of Fourier modes $\{\vec{q}_i\}$ minimizing the interaction kernel) need not be continuous as a function of the parameters that define the interactions. As we will simply illustrate below, in a manner that is mathematically similar to that appearing in the Ginzburg-Landau

constructs, a "first order transition" in the value of the ground state modulation lengths can arise. Such a possibility is quite obvious and need not be expanded upon in depth. As an illustrative example, let us consider the Range=3 interaction kernel

$$v(k) = a[\Delta+\epsilon] + \frac{1}{2}b[\Delta+\epsilon]^2 + \frac{1}{3}c[\Delta+\epsilon]^3, \qquad (74)$$

with $[0 < \epsilon \ll 1]$ and $c > 0$. If the parameters are such that $a > 0$ and $b < 0$, then $v(k)$ displays three minima, i.e. $[\Delta+\epsilon] = 0$ and $[\Delta+\epsilon] = \pm m_+^2$, where $m_+^2 = \frac{1}{2c}[-b+\sqrt{b^2-4ac}]$. the locus of points in the ab plane where the three minima are equal to one another is defined by $v(k) = 0$. This leads to the relation $m_+^2 = -\frac{4a}{b}$. Putting all of the pieces together, we see that $b = -4\sqrt{ca/3}$ constitutes a line of "first order transitions". On traversing this line of "first order transitions", the minimizing $[\Delta+\epsilon]$ (and thus the minimizing wavenumbers) changes discontinuously by $\Delta m = (-\frac{4a}{b})^{1/2} = (\frac{3a}{c})^{1/4}$.

VII. EXAMPLE SYSTEMS

In this section, we will investigate in detail several frustrated systems. We will start our analysis by examining the screened Coulomb Frustrated Ferromagnet. A screened Coulomb interaction of screening length λ^{-1} has the continuum Fourier transformed interaction kernel $v(k) = [k^2+\lambda^2]^{-1}$. The lowest order non-vanishing derivative of $v_L(k)$ of order higher than two is that of $p = 3$. Invoking Eq.(47), we find a modulation length that increases with decreasing temperature as $T \to T_c^+$ (see also appendix B, Eq. (B1) in particular) .

The dipolar interaction can be thought of as the $\delta \to 0$ limit of the interaction,

$$V_d = \frac{1}{[(\vec{x}-\vec{y})^2+\delta^2]^{3/2}}. \qquad (75)$$

This form has a simple Fourier transform. In two spatial dimensions,

$$v_d(k) = 2\pi\delta^{-1}e^{-k\delta}. \qquad (76)$$

In three dimensions,

$$v_d(k) = 4\pi K_0(k\delta), \qquad (77)$$

In this case, we similarly find that the first non-vanishing derivative of v_L is order of order $p = 3$ in the notation of Eq.(47). This, as well as the detailed form of Eq.(B1) suggest an increasing modulation length with decreasing temperature as $T \to T_c^+$.

A. Numerical evaluation of the Correlation function

In Figs.(5,6), we display a numerical evaluation of the correlation function for the Coulomb frustrated ferromagnet and the dipolar frustrated ferromagnet (see Eqs.(5, 6, 8)) in a two dimensional region of size 100×100. In both systems, we see that for low temperatures, the modulation length increases with temperature, thus satisfying our criterion of Eq.(47) [see also appendix B, Eq. (B1) in particular].

B. Coexisting short range and screened Coulomb interactions

In this section, we study the screened Coulomb frustrated ferromagnet in more details. The Fourier transform of the interaction kernel of Eq.(4) is

$$v(k) = Jk^2 + \frac{Q}{k^2+\lambda^2}. \qquad (78)$$

In Appendix (C), we provide explicit expressions for the dependence of μ on the temperature T. This dependence delineates the different temperature regimes. For $T > T^*$ wherein the temperature T^* is set by $\mu(T^*) = J\lambda^2 + 2\sqrt{JQ}$), the pair correlator in $d = 3$ dimensions is given by

$$G(\vec{x}) = \frac{k_BT}{4\pi J|\vec{x}|}\frac{1}{\beta^2-\alpha^2} \times[e^{-\alpha|\vec{x}|}(\lambda^2-\alpha^2) - e^{-\beta|\vec{x}|}(\lambda^2-\beta^2)]. \qquad (79)$$

Here,

$$\alpha^2, \beta^2 = \frac{\lambda^2+\mu/J \mp \sqrt{(\lambda^2-\mu/J)^2-4Q/J}}{2}. \qquad (80)$$

By contrast, for temperatures $T < T^*$, we obtain an analytic continuation of Eq.(79) to complex α and β,

$$G(\vec{x}) = \frac{k_BT}{8\alpha_1\alpha_2\pi J|\vec{x}|}e^{-\alpha_1|\vec{x}|} \times[(\lambda^2-\alpha_1^2+\alpha_2^2)\sin\alpha_2|\vec{x}| + 2\alpha_1\alpha_2\cos\alpha_2|\vec{x}|],. \qquad (81)$$

In Eq.(81), $\alpha = \alpha_1 + i\alpha_2 = \beta^*$. In a similar fashion, in $d = 2$ spatial dimensions, for $T > T^*$,

$$G(\vec{x}) = \frac{k_BT}{2\pi}\frac{1}{\beta^2-\alpha^2}[(\lambda^2-\alpha^2)K_0(\alpha|\vec{x}|) +(\beta^2-\lambda^2)K_0(\beta|\vec{x}|)]. \qquad (82)$$

As in the three dimensional case, the high temperature correlator of Eq.(82) may be analytically continued to lower temperatures, $T < T^*$, for which α and β become complex.

High temperature limit

In the high temperature limit, in two spatial dimensions,

$$G(\vec{x}) = \frac{k_BT}{2\pi J}K_0\left(\sqrt{\frac{k_BT\Lambda^2}{4\pi J}}|\vec{x}|\right) - \frac{8\pi}{k_BT\Lambda^4}K_0(\lambda|\vec{x}|). \qquad (83)$$

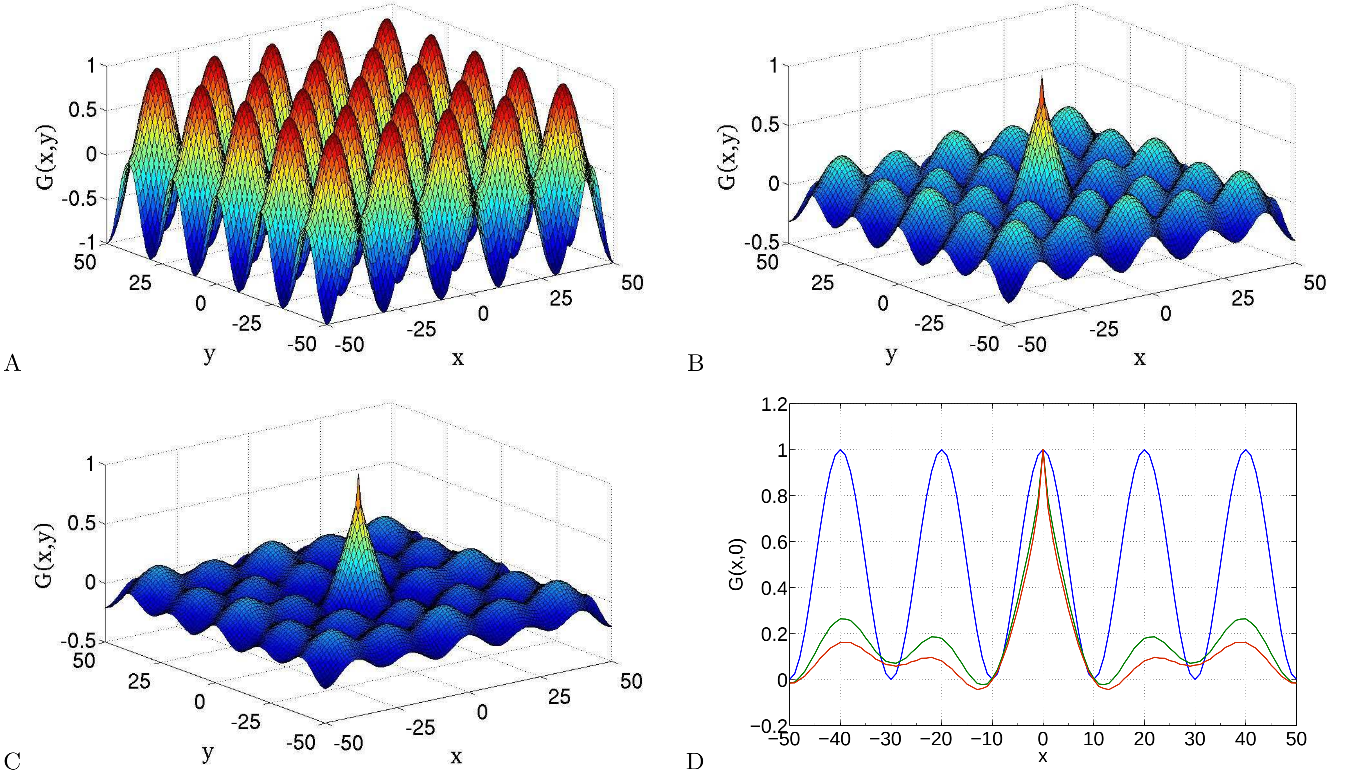

FIG. 5: The correlator $G(x, y)$ for a two dimensional Screened Coulomb Ferromagnet of size 100 by 100. $J = 1$, $Q = 0.0004$, Screening length= $100\sqrt{2}$. A: $\mu = \mu_{min} = -0.0874$, B: $\mu = \mu_{min} + 0.001$, C: $\mu = \mu_{min} + 0.003$, D: G(x,y) for $y = 0$ for A(blue)[$L_D = 20$], B(green)[$L_D = 24$] and C(red)[$L_D = 26$].

In three spatial dimensions,

$$G(\vec{x}) = \frac{k_B T}{4\pi J|\vec{x}|} e^{-\sqrt{\frac{k_B T \Lambda^3}{6\pi^2 J}}|\vec{x}|} - \frac{9\pi^3 Q}{k_B T \Lambda^6 |\vec{x}|} e^{-\lambda|\vec{x}|}. \quad (84)$$

In the unscreened case, in two spatial dimensions,

$$G(\vec{x}) = \frac{k_B T}{2\pi J} K_0\left(\sqrt{\frac{k_B T \Lambda^2}{4\pi J}}|\vec{x}|\right) - \frac{8\pi}{k_B T \Lambda^4} K_0\left(\sqrt{\frac{4\pi Q}{k_B T \Lambda^2}}|\vec{x}|\right). \quad (85)$$

In three spatial dimensions,

$$G(\vec{x}) = \Big[\frac{k_B T}{4\pi J|\vec{x}|} e^{-\sqrt{\frac{k_B T \Lambda^3}{6\pi^2 J}}|\vec{x}|} - \frac{9\pi^3 Q}{k_B T \Lambda^6 |\vec{x}|} e^{-\sqrt{\frac{6\pi^2 Q}{k_B T \Lambda^3}}|\vec{x}|}\Big]. \quad (86)$$

From the above expressions, it is clear that the coefficients of the terms corresponding to the diverging correlation length goes to zero in the high temperature limit.

We note that two correlation lengths are manifest for all $(\mu - J\lambda^2)^2 > 4JQ$. This includes all unfrustrated screened attractive Coulomb ferromagnets (those with $Q < 0$)). The evolution of the correlation functions may be traced by examining the dynamics of the poles in the complex k plane as a function of temperature. At high temperatures, correlations are borne by poles that lie on the imaginary k axis.

Thermal evolution of modulation length at low temperatures

At $T = T^*$, the poles merge in pairs at $k = \pm i\sqrt{\lambda^2 + \sqrt{Q/J}}$. At lower temperatures, $T < T^*$, the poles move off the imaginary axis (leading in turn to oscillations in the correlation functions). The norm of the poles, $|\alpha| = (Q/J + \lambda^2\mu(T)/J)^{1/4}$ tends to a constant in the limit of vanishing screening ($\lambda^{-1} = 0$) wherein the after merging at $T = T^*$, the poles slide along a circle [Fig.(7)]. In the low temperature limit of the unscreened Coulomb ferromagnet, the poles hit the real axis at finite k, reflecting oscillatory modulations in the ground state. In the presence of screening, the pole trajectories are slightly skewed [Fig.(8)] yet for $Q/J > \lambda^4$, α tends to the ground state modulation wavenumber $\sqrt{\sqrt{Q/J} - \lambda^2}$. If the screening is sufficiently large, i.e., if the screening

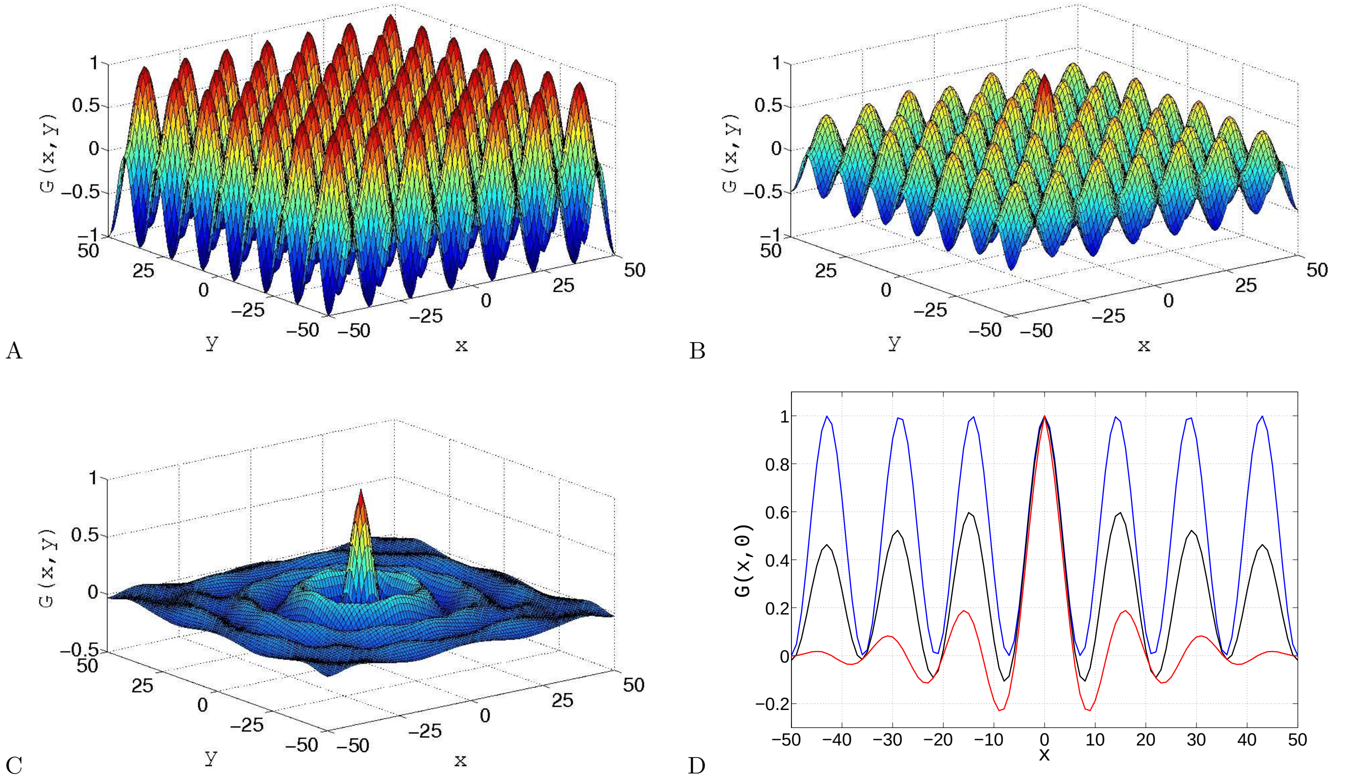

FIG. 6: The correlator $G(x, y)$ for a two dimensional Dipolar Ferromagnet of size 100 by 100. $J = 1$, $Q = 0.15$. A: $\mu = \mu_{min} = -1.1459$, B: $\mu = \mu_{min} + 4 \times 10^{-5}$, C: $\mu = \mu_{min} + 1 \times 10^{-3}$, D: G(x,y) for $y = 0$ for A(blue)[$L_D = 14$], B(green)[$L_D = 15$] and C(red)[$L_D = 16$].

length is shorter than the natural period favored by a balance between the unscreened Coulomb interaction and the nearest neighbor attraction ($\lambda > (Q/J)^{1/4}$), then the correlation functions never exhibit oscillations. In such instances, the poles continuously stay on the imaginary axis and, at low temperatures, one pair of poles veers towards $k = 0$ reflecting the uniform ground state of the heavily screened system.

To summarize, at high temperatures the pair correlator $G(x)$ is a sum of two decaying exponentials (one of which has a correlation length which diverges in the high temperature limit). For $T < T^*$ in under-screened systems, one of the correlation lengths turns into a modulation length characterizing low temperature oscillations. At the cross-over temperature T^*, the modulation length is infinite. As the temperature is progressively lowered, the modulation length decreases in size – until it reaches its ground state value. The temperature $T^*(Q/J, \lambda)$ is a "disorder line" [33] like temperature [Fig(9)]. An analytical thermodynamic crossover does occur at $T = T^*$. A large n calculation illustrates that the internal energy per particle

$$\frac{U}{N} = \frac{1}{2}(k_B T - \mu), \tag{87}$$

To detect a crossover in U and that in other thermodynamic functions, the forms of μ both above and below T^* may be derived from the spherical model normalization condition to find that the real valued functional form of $\mu(T)$ changes [See appendix C].

The system starts to exhibit order at the critical temperature $T = T_c$ given by

$$\frac{1}{k_B T_c} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{v(\vec{k}) - v(\vec{q})}. \tag{88}$$

For $Q/J > \lambda^4$, the modulus of the minimizing (ground state) wavenumber ($|\vec{q}|$) is given by

$$q = \frac{2\pi}{L_D^g} = \sqrt{\sqrt{Q/J} - \lambda^2}, \tag{89}$$

with L_D^g the ground state modulation length. Associated with this wavenumber is the kernel $v(\vec{q}) = 2\sqrt{JQ} - J\lambda^2$ to be inserted in Eq.(88) for an evaluation of the critical temperature T_c. Similarly, the ground state wavenumber $\vec{q} = 0$ whenever $Q/J < \lambda^4$. If $Q/J > \lambda^4$ and modulations transpire for temperatures $T < T^*$, the critical temperature at which the chemical potential of Eq.(25), $\mu(T_c) = J\lambda^2 - 2\sqrt{JQ}$, is lower than the crossover temperature T^* (given by $\mu(T^*) = J\lambda^2 + 2\sqrt{JQ}$) at which modulations first start to appear. The Screened Coulomb

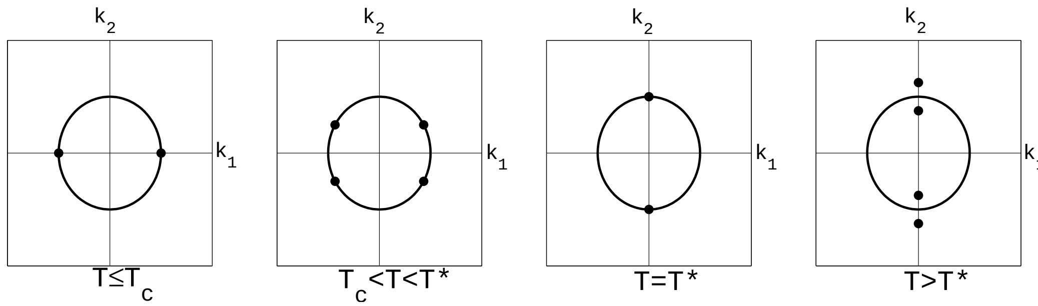

FIG. 7: Location of the poles with increasing temperature (left to right) in the complex k-plane for the Coulomb frustrated ferromagnet. For temperatures below T_c, all the poles are real. Above T_c, the poles split in opposite directions of the real axis to give rise to two new complex poles. For $T_c < T < T^*$, we have complex poles. At T^*, pairs of such poles meet on the imaginary axis. Above T^*, the poles split along the imaginary axis. Thus, above T^*, the poles are purely imaginary.

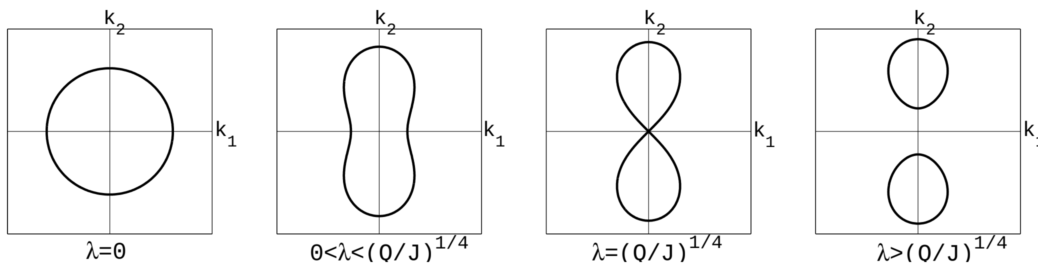

FIG. 8: Trajectory of the poles in the complex k-plane for $T_c < T < T^*$ for the screened Coulomb ferromagnet. The screening length, λ^{-1} decreases from left to right. In the first figure $\lambda = 0$ and $\lambda > (Q/J)^{1/4}$ in the last figure.

Ferromagnet has $T_c(Q/J = \lambda^4) > 0$ in $d > 4$ dimensions and in any dimension $T_c(Q/J > \lambda^4) = 0$. For small finite n, a first order Brazovskii transition may replace the continuous transition occurring at T_c within the large n limit [34]. Depending on parameter values such an equilibrium transition may or may not transpire before a possible glass transition may occur [22].

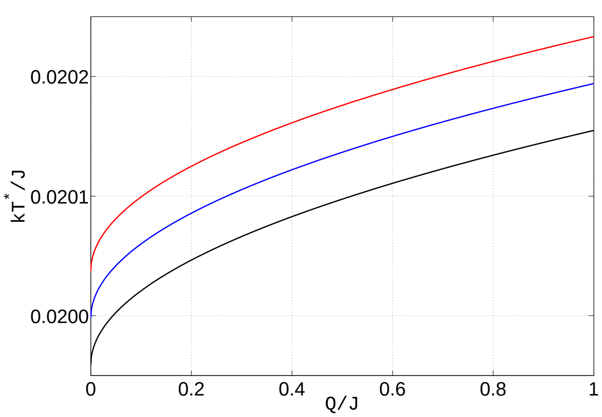

FIG. 9: Temperature at which the modulation length diverges for a 100×100 Coulomb frustrated ferromagnet plotted versus the relative strength of the Coulomb interaction with respect to the ferromagnetic interaction. [Blue:$\lambda = \lambda_0 = 1/(100\sqrt{2})$; Red:$\lambda = 0.999\lambda_0$; Black:$\lambda = 1.001\lambda_0$]

Domain length scaling in the Coulomb Frustrated Ferromagnet

The characteristic length scales are governed by the position of the poles of $[v(k) + \mu]^{-1}$. See Fig.(7) for an illustration of the pole locations at low temperatures. For the frustrated Coulomb ferromagnet of Eq.(78) in the absence of screening ($\lambda^{-1} = 0$),

$$v(k) + \mu = \frac{J}{k^2}(k^4 + \frac{\mu}{J}k^2 + \frac{Q}{J}). \qquad (90)$$

Eq.(90) enable us to determine, in our large n analysis, the cross-over temperature T^* at which $\mu^* = \mu(T^*) = 2\sqrt{JQ}$. At $T = T^*$, the poles lie on the imaginary axis in k-plane. As the temperature is lowered below T^*, the two poles bifurcate. This bifurcation gives rise to finite size spatial modulations. At temperatures $T < T^*$, the four poles slide along a circle of fixed radius of size $(Q/J)^{1/4}$ (see Fig.7). At zero temperature, these four poles merge in pairs to form two poles that lie on the real axis. The inverse modulation length is set by the absolute values of the real parts of the poles. We will set $\mu \equiv (2\sqrt{JQ} - \delta\mu)$.

In the following, we will obtain the dependence of the real part of the poles on $\delta\mu$. The poles of $1/(v(k)+\mu)$ are determined by

$$k_{pole}^2 = -\frac{\mu}{2J} \pm i\sqrt{\frac{Q}{J} - \frac{\mu^2}{4J^2}} = \sqrt{\frac{Q}{J}}e^{\pm 2i\theta}. \quad (91)$$

At $\mu = \mu^*$, the angle $\theta = \pi/2$. This point corresponds to the transition between (i) the high temperature region ($T > T^*$) wherein the system does not exhibit any modulations and (ii) the low temperature region ($T < T^*$). [See Fig.(7).] Eq.(91) implies that $\cos 2\theta = [1 - \frac{\delta\mu}{\mu^*}]$ or

$$k_{pole,real} = \frac{\delta\mu^{1/2}}{2J^{1/2}}, \quad (92)$$

in accord with the general result of Eq.(17).

C. Full direction and location dependent dipole-dipole interactions

In this subsection and the next, we consider systems where the spins are three dimensional and the interactions have the appropriate directional dependence. In this subsection, we will consider the effect of including the full dipolar interactions vis a vis the more commonly used scalar product form between two dipoles that is pertinent to two dimensional realizations. The dipolar interaction is given by

$$H_{dip} = \sum_{\vec{x}\neq\vec{y}} \left[\frac{\vec{S}(\vec{x})\cdot\vec{S}(\vec{y})}{r^3} - \frac{3[\vec{S}(\vec{x})\cdot\vec{r}][\vec{S}(\vec{y})\cdot\vec{r}]}{r^5}\right]. \quad (93)$$

The two point correlator for a ferromagnetic system frustrated by this interaction is given, in the large n approximation, by

$$G(\vec{x}) = k_B T \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k}\cdot\vec{x}} \left[\frac{2}{J\Delta(\vec{k}) + Qv_d(k) + \mu} + \frac{1}{J\Delta(\vec{k}) - 2Qv_d(k) + \mu}\right], (94)$$

where $v_d(k)$ is given by Eqs.(76,77). For temperatures $T \leq T_c$,

$$\mu_{min} = -\min_{k\in\mathbb{R}}[J\Delta(\vec{k}) + Qv_d(k), J\Delta(\vec{k}) - 2Qv_d(k)]. (95)$$

The Fourier transformed dipolar interaction kernel is positive definite, $v_d(k) > 0$. An unscreened dipolar interaction leads to a $v_d(k)$ that diverges (tends to negative infinity) at its minimum at $k = 0$. In the presence of both upper and lower distance cutoffs (see, e.g., Eq.(8) for a lower cutoff) on the dipolar interaction, the minimum of $v_d(k)$ attains a finite value and the system has a finite critical temperature.

Examining Eq.(94), we see that the introduction of the angular dependence in the dipolar interaction changes the results that would be obtained if the angular dependence were not included in a dramatic way.

(i) New correlation and modulation lengths arise from the second term in Eq.(94).

(ii) At low temperatures, the second term in Eq.(94) becomes dominant as its poles have a smaller real part (and thus a larger correlation length) relative to the first term in Eq.(94) that appears for an isotropic dipole-dipole interactions.

D. Dzyaloshinsky- Moriya Interactions

As another example of a system with interactions having non-trivial directional dependence, we consider a system of three component spins with the Dzyaloshinsky-Moriya interaction [35] present along with the ferromagnetic interaction and a long range interaction,

$$H = -J\sum_{\langle\vec{x},\vec{y}\rangle} \vec{S}(\vec{x})\cdot\vec{S}(\vec{y}) + \sum_{\langle\vec{x},\vec{y}\rangle} \vec{D}\cdot[\vec{S}(\vec{x})\times\vec{S}(\vec{y})] + Q\sum_{\vec{x}\neq\vec{y}} V_L(|\vec{x}-\vec{y}|)\vec{S}(\vec{x})\cdot\vec{S}(\vec{y}). \quad (96)$$

We diagonalize this interaction kernel to obtain a Hamiltonian of the form,

$$H = \sum_{\vec{x},\vec{y}}\sum_a \hat{S}_a^*(\vec{x})V_a(\vec{x},\vec{y})\hat{S}_a(\vec{y}). \quad (97)$$

The $\hat{S}_a$'s are linear combinations of the components of $\vec{S}$. In a large n approximation, the two point correlator is given by

$$G(\vec{x}) = k_B T \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k}\cdot\vec{x}} \left[\frac{1}{J\Delta(\vec{k}) + Qv_L(k) + \mu} + \frac{2(J\Delta(\vec{k}) + Qv_L(k) + \mu)}{(J\Delta(\vec{k}) + Qv_L(k) + \mu)^2 + (D_1^2 + D_2^2 + D_3^2)[\Delta(\vec{k})]^2}\right] (98)$$

The presence of the Dzyaloshinsky-Moriya interaction does not alter the original poles and hence does not change the original lengthscales of the system. However, additional lengthscales arise due to the second term in Eq.(98).

VIII. CONCLUSIONS

In conclusion, we

(1) showed the existence of certain crossover temperatures at which the lengthscales of the system change character. As the long range interaction strength is turned off, the crossover temperature tends to the critical temperature of the unfrustrated ferromagnetic system. The

divergence of the modulation length at the crossover temperature defines a new critical exponent ν_L. We proved that, quite universally, $\nu_L = 1/2$.

(2) studied the evolution of the ground state modulation lengths in frustrated Ising systems.

(3) discovered a divergence of correlation lengths in the high temperature limit of long range large n systems. This divergence is replaced by a saturation when the long range interactions have are screened. Notwithstanding this divergence, taking note of the prefactors associated with these correlation length, the amplitude of the correlations themselves decreases with increasing temperatures.

(4) investigated, in large n theories, the evolution of modulation and correlation lengths as a function of temperature in different classes of systems.

(5) proved that, in large n theories, the combined sum of the number of correlation and the number of modulation lengths is conserved. We have also showed that there exists a diverging modulation length at high temperatures for systems with long range interactions.

(6) studied three dimensional dipolar systems. We found that the full dipolar interactions with angular dependence included, changes the ground state of the system and also adds new lengthscales.

It is a pleasure to acknowledge many discussions with Lincoln Chayes, Daniel Kivelson, Steven Kivelson, Joseph Rudnick, Gilles Tarjus, and Peter Wolynes. We also thank the Center for Materials Innovation – Washington University in St. Louis for providing partial financial support for this project.

APPENDIX A: TRANSFER MATRIX IN THE ONE-DIMENSIONAL SYSTEM WITH ISING SPINS

Thus far, we focused primarily on high dimensional continuous spin systems. For completeness, we review and illustrate how some similar conclusions can be drawn for one dimensional Ising systems with finite ranged interactions and briefly discuss trivial generalizations. In particular, we show how the sum of the number of modulation and number of correlation lengths does not change as the temperature is varied. In Section(VI C 3), we illustrated how this arises for general large n systems.

For interactions of range R in a one dimensional Ising spin chain, the transfer matrix, T is of dimension $M = \mathcal{O}(2^R)$. The correlation function for large system size, takes the form

$$G(x) = \sum_{k=1}^{2^R-1} A_k \left(\frac{\lambda_k}{\lambda_0}\right)^x, \tag{A1}$$

where λ_is are the eigenvalues of the transfer matrix. Since the characteristic equation has real non-negative coefficients, from Perron-Frobenius theorem, λ_0 is real, positive and is non-degenerate. The secular equation, $\det(T - \lambda I) = 0$ is a polynomial in λ with real coefficients. Thus, two possibilities need to be examined: real roots, and, complex conjugate pairs of roots. Real eigenvalues λ_p give terms with correlation length,

$$\xi = \ln\left(\frac{\lambda_0}{|\lambda_p|}\right). \tag{A2}$$

Complex conjugate eigenvalues, λ_q and λ_q^* correspond to the same correlation length and modulation length, given by,

$$\xi = \ln\left(\frac{\lambda_0}{|\lambda_q|}\right), \tag{A3}$$

$$L_D = \frac{2\pi}{\tan^{-1}\left(\frac{Im\lambda_q}{Re\lambda_q}\right)}. \tag{A4}$$

Thus, the total number of correlation and modulation lengths is the order of the polynomial in λ in the secular equation, or simply the dimension of the transfer matrix – $\mathcal{O}(2^R)$. Similar to our conclusions for the high dimensional continuous spin systems, this number does not vary with temperature. For q state Potts type spins, replicating the above arguments mutatis mutandis, we find that the total number of correlation and modulation lengths is $\mathcal{O}(q^R)$. Similarly, for such a system placed on a d dimensional slab of finite extent in, at least, $(d-1)$ directions along which it has a length of order $\mathcal{O}(l) > R$, there will be $\mathcal{O}(q^l)$ transfer matrix eigenvalues and thus an identical number for the sum of the number of modulation lengths with the number of correlation lengths.

The eigenvalues change from being complex below certain crossover temperatures to being purely real above. These temperatures form the "disorder line".

APPENDIX B: DETAILED EXPRESSIONS FOR δL_D FOR DIFFERENT ORDERS $p > 2$ AT WHICH THE LONG RANGE INTERACTION KERNEL HAS ITS FIRST NON-VANISHING DERIVATIVE

If the lowest order non-vanishing derivative of $v_L(k)$ at the minimizing wavenumber k_0 (see Eq.(37)). of order higher than two is that with $p = 3$ (the general case) then, in the large n limit, the change in the modulation length at temperatures $T > T_c$ about its value at $T = T_c$ of Eq.(47) is given by

$$\delta L_D = -\frac{2\pi}{k_0^2} \frac{Q v_L^{(3)}(k_0)\delta\mu}{12(J + \frac{Q v_L^{(2)}(k_0)}{2!})^2}. \tag{B1}$$

We employ Eq.(B1) in our analysis in Section(VII). If the lowest order derivatives are of order $p = 4$ or 5 then,

$$\delta L_D = \frac{2\pi}{k_0^2} \frac{Q v_L^{(5)}(k_0)(\delta\mu)^2}{240(J + \frac{Q v_L^{(2)}(k_0)}{2!})^3}. \tag{B2}$$

Similarly, for $p = 6$ or 7,

$$\delta L_D = -\frac{2\pi}{k_0^2}\frac{Q v_L^{(7)}(k_0)(\delta\mu)^3}{10080(J + \frac{Q v_L^{(2)}(k_0)}{2!})^4}, \tag{B3}$$

and so on.

APPENDIX C: $\mu(T)$ FOR THE SCREENED COULOMB FERROMAGNET

In three dimensions, with $\Lambda = 2\pi/a$ an ultra-violet cutoff with a the lattice unit length, at high temperatures, $T > T^*$, this leads to the following implicit equation for $\mu(T)$ in the case of the screened Coulomb ferromagnet of Eq.(78),

$$\frac{1}{T} = \frac{\Lambda}{2\pi^2} + \frac{\sqrt{2}}{4\pi^2 p} \times\Big(\frac{\lambda^2\mu - \mu^2 + \mu p - 2Q}{\sqrt{\lambda^2 + \mu + p}}\tan^{-1}(\frac{\Lambda\sqrt{2}}{\sqrt{\lambda^2+\mu+p}}) - \frac{\lambda^2\mu - \mu^2 + \mu p + 2Q}{\sqrt{\lambda^2 + \mu - p}}\tan^{-1}(\frac{\Lambda\sqrt{2}}{\sqrt{\lambda^2+\mu-p}})\Big) \tag{C1}$$

Here, we employed the shorthand $p \equiv \sqrt{(\mu - \lambda^2)^2 - 4Q}$. This parameter p vanishes at the crossover temperature T^* at which a divergent modulation length makes an appearance, $p(T = T^*) = 0$. At low temperatures, $T < T^*$, p becomes imaginary and an analytical crossover occurs to another real functional form.

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